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# Large-time behavior of spherically symmetric flow for viscous heat-conductive gas

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## 1 Introduction

We consider the asymptotic behavior of a spherically symmetric solution to a polytropic ideal model of a compressible viscous gas over an unbounded exterior domain $\Omega := \{\xi \in \mathbb{R}^n ; |\xi| > 1\}$, where $n$ $(\geq 3)$ is a space dimension. The motion of the polytropic ideal gas is governed by the system of equations

$$\rho_t + \nabla\cdot(\rho u) = 0, \tag{1.1a}$$

$$\rho\{u_t + (u\cdot\nabla)u\} = \mu_1\Delta u + (\mu_1+\mu_2)\nabla(\nabla\cdot u) - \nabla P(\rho,\theta) + \rho f, \tag{1.1b}$$

$$c_V\rho\{\theta_t + (u\cdot\nabla)\theta\} = \kappa\Delta\theta - P(\rho,\theta)\nabla\cdot u + \mu_2(\nabla\cdot u)^2 + 2\mu_1 D\cdot D \tag{1.1c}$$

in the Eulerian coordinate, where the mass density $\rho$, the velocity $u = (u_1, \ldots, u_n)$ and the absolute temperature $\theta$ are unknown functions. In addition, $P(\rho,\theta) = R\rho\theta$ $(R > 0)$ is the pressure; $f = f(\xi)$ is the external force; $\mu_1$ and $\mu_2$ are constants called the viscosity coefficients satisfying $\mu_1 > 0$ and $2\mu_1 + n\mu_2 > 0$; $\kappa$ and $c_V$ are positive constants called the thermal conductivity and the specific heat at constant volume, respectively; $D = D(u)$ is the deformation tensor,

$$D\cdot D := \sum_{i,j=1}^{n} D_{ij}^2 \quad \text{and} \quad D_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i}\right).$$

Our concern is the problem of which initial data $(\rho, u, \theta)(\xi, 0) = (\rho_0, u_0, \theta_0)(\xi)$ is given by the spherically symmetric function:

$$\rho_0(\xi) = \hat\rho_0(r), \quad u_0(\xi) = \frac{\xi}{r}\hat u_0(r), \quad \theta_0(\xi) = \hat\theta_0(r), \tag{1.2}$$

where $r := |\xi|$ and each $\hat\rho_0$, $\hat u_0$ and $\hat\theta_0$ is a scaler function. We assume that the external force $f$ is also given by the spherically symmetric potential force,

$$f := -\nabla U = -\frac{\xi}{r}U_r(r). \tag{1.3}$$

Under the assumptions (1.2) and (1.3), the solution $(\rho, u, \theta)$ to (1.1) becomes spherically symmetric since (1.1) is rotationally invariant (see [6]). Namely, a solution to (1.1) is in the form of

$$\rho(\xi,t) = \hat\rho(r,t), \quad u(\xi,t) = \frac{\xi}{r}\hat u(r,t), \quad \theta(\xi,t) = \hat\theta(r,t). \tag{1.4}$$

For simplicity, hereafter, we abbreviate the symbol "^" to express spherically symmetric functions. Therefore, the equations for the spherically symmetric solution $(\rho, u, \theta)$ is

$$\rho_t + \frac{(r^{n-1}\rho u)_r}{r^{n-1}} = 0, \tag{1.5a}$$

$$\rho(u_t + uu_r) = \mu\left(\frac{(r^{n-1}u)_r}{r^{n-1}}\right)_r - P(\rho,\theta)_r - \rho U_r, \tag{1.5b}$$

$$c_V\rho(\theta_t + u\theta_r) = \kappa\frac{(r^{n-1}\theta_r)_r}{r^{n-1}} - P(\rho,\theta)\frac{(r^{n-1}u)_r}{r^{n-1}} + \mu_2\left(\frac{(r^{n-1}u)_r}{r^{n-1}}\right)^2 + 2\mu_1 u_r^2 + 2(n-1)\mu_1\frac{u^2}{r^2}, \tag{1.5c}$$

where $\mu := 2\mu_1 + \mu_2$ is a positive constant. The initial and the boundary conditions are prescribed as

$$\rho(r,0) = \rho_0(r), \quad u(r,0) = u_0(r), \quad \theta(r,0) = \theta_0(r), \tag{1.6}$$

$$u(1,t) = 0, \quad \theta_r(1,t) = 0. \tag{1.7}$$

It is also assumed that the initial data satisfies

$$\inf_{r\in[1,\infty)} \rho_0(r) > 0, \qquad \inf_{r\in[1,\infty)} \theta_0(r) > 0, \tag{1.8}$$

$$\lim_{r\to\infty}(\rho_0(r), u_0(r), \theta_0(r)) = (\rho_+, u_+, \theta_+), \quad \rho_+ > 0, \quad \theta_+ > 0, \tag{1.9}$$

where $\rho_+, u_+$ and $\theta_+$ are constants. Moreover, the initial data $(\rho_0, u_0, \theta_0)$ is supposed to be compatible with the boundary data (1.7):

$$u_0(1) = 0, \quad \theta_{0r}(1) = 0, \tag{1.10a}$$

$$\left\{\mu\left(\frac{(r^{n-1}u_0)_r}{r^{n-1}}\right)_r - P(\rho_0,\theta_0)_r - \rho_0 U_r\right\}\Bigg|_{r=1} = 0. \tag{1.10b}$$

The aim of the present paper is to show that the solution to the problem (1.5), (1.6) and (1.7) converges to the corresponding stationary solution as time goes to infinity for an arbitrary initial disturbance belonging to $H^1$ Sobolev space. The stationary solution $(\tilde\rho(r), \tilde u(r), \tilde\theta(r))$ is a solution to the equations (1.5) independent of time $t$ and satisfies the same boundary and spatial asymptotic conditions (1.7) and (1.9). Therefore, the stationary solution verifies

$$\frac{(r^{n-1}\tilde\rho\tilde u)_r}{r^{n-1}} = 0, \tag{1.11a}$$

$$\tilde\rho\tilde u\tilde u_r = \mu\left(\frac{(r^{n-1}\tilde u)_r}{r^{n-1}}\right)_r - P(\tilde\rho,\tilde\theta)_r - \tilde\rho U_r, \tag{1.11b}$$

$$c_V\tilde\rho\tilde u\tilde\theta_r = \kappa\frac{(r^{n-1}\tilde\theta_r)_r}{r^{n-1}} - P(\tilde\rho,\tilde\theta)\frac{(r^{n-1}\tilde u)_r}{r^{n-1}} + \mu_2\left(\frac{(r^{n-1}\tilde u)_r}{r^{n-1}}\right)^2 + 2\mu_1\tilde u_r^2 + 2(n-1)\mu_1\frac{\tilde u^2}{r^2}, \tag{1.11c}$$

$$\tilde{u}(1) = 0, \quad \tilde{\theta}_r(1) = 0, \quad \lim_{r\to\infty}(\tilde{\rho}(r), \tilde{u}(r), \tilde{\theta}(r)) = (\rho_+, u_+, \theta_+). \tag{1.12}$$

Solving the problem (1.11) with (1.12), we obtain the stationary solution explicitly as

$$\tilde{\rho}(r) = \rho_+ \exp\left(-\frac{1}{R\theta_+}U(r)\right), \quad \tilde{u}(r) = 0, \quad \tilde{\theta}(r) = \theta_+, \tag{1.13}$$

where we have assumed that

$$\lim_{r\to\infty} U(r) = \lim_{r\to\infty} \int_1^r U_r(\eta)\, d\eta + U(1) = 0 \tag{1.14}$$

without loss of generality. Here, let us note that the stationary solution is a constant $(\rho_+, 0, \theta_+)$ if the external force is equal to zero, that is, $U_r(r) \equiv 0$.

The main result in the present paper is stated in the next theorem, which means that the stationary solution (1.13) is time asymptotically stable. This stability theorem does not need any smallness assumptions on the initial data. Moreover, due to the condition (1.16), if the external force $U_r$ is attractive, i.e., $U_r(r) \geq 0$, it can be arbitrary large. The typical example allowing this assumption is the case that the external force is given by a gravitational force of the sphere.

**Theorem 1.1.** *Suppose that the initial data $(\rho_0, u_0, \theta_0)$ satisfies*

$$r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho}),\ r^{\frac{n-1}{2}}u_0,\ r^{\frac{n-1}{2}}(\theta_0 - \theta_+),\ r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho})_r,\ r^{\frac{n-1}{2}}u_{0r},\ r^{\frac{n-1}{2}}\theta_{0r} \in L^2(1,\infty), \tag{1.15a}$$

$$\rho_0 \in \mathcal{B}^{1+\sigma}[1,\infty),\ (u_0, \theta_0) \in \mathcal{B}^{2+\sigma}[1,\infty) \tag{1.15b}$$

*for a certain constant $\sigma \in (0,1)$. Then there exists a constant $\delta > 0$ such that if the external force $U_r \in C^1[1,\infty) \cap \mathcal{B}^\sigma[1,\infty)$ satisfies*

$$-\delta \leq U_r(r) \tag{1.16}$$

*for an arbitrary $r \geq 1$, then the initial boundary value problem* (1.5), (1.6) *and* (1.7) *has a unique solution $(\rho, u, \theta)$ satisfying*

$$r^{\frac{n-1}{2}}(\rho - \tilde{\rho}),\ r^{\frac{n-1}{2}}u,\ r^{\frac{n-1}{2}}(\theta - \theta_+),\ r^{\frac{n-1}{2}}(\rho - \tilde{\rho})_r,\ r^{\frac{n-1}{2}}u_r,\ r^{\frac{n-1}{2}}\theta_r \in C([0,T]; L^2(1,\infty)), \tag{1.17a}$$

$$\rho \in \mathcal{B}^{1+\sigma,1+\sigma/2}([1,\infty)\times[0,T]),\ (u,\theta) \in \mathcal{B}^{2+\sigma,1+\sigma/2}([1,\infty)\times[0,T]) \tag{1.17b}$$

*for an arbitrary $T > 0$. Moreover, the solution $(\rho, u, \theta)$ converges to the corresponding stationary solution* (1.13) *as time tends to infinity:*

$$\lim_{t\to\infty} \sup_{r\in[1,\infty)} |(\rho(r,t) - \tilde{\rho}(r), u(r,t), \theta(r,t) - \theta_+)| = 0. \tag{1.18}$$

**Remark 1.2.** The same result as in Theorem 1.1 is proved for the case $P(\rho,\theta) = R\rho^\alpha\theta$ $(\alpha \geq 1)$ in place of $P(\rho,\theta) = R\rho\theta$. It is slightly more general than the ideal polytropic gas. If $\alpha > 1$, the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ is given by

$$\tilde{\rho}(r) = \left(\rho_+^{\alpha-1} + \frac{\alpha-1}{\alpha R\theta_+}U(r)\right)^{1/(\alpha-1)}, \quad \tilde{u}(r) = 0, \quad \tilde{\theta}(r) = \theta_+,$$

which is proved similarly as the derivation of (1.13). The difference of the proofs of the stabilities for the cases $\alpha = 1$ and $\alpha > 1$ appears in the derivation of the upper bound of the specific volume.

**Remark 1.3.** For the Dirichlet boundary condition on the temperature, the same result as in Theorem 1.1 holds if $\theta(1,t) = \theta_+$ is posed in place of $\theta_r(1,t) = 0$ in (1.7).

**Related results.** The asymptotic behavior of a solution to the compressible Navier-Stokes equation (1.1) is first considered by Matsumura and Nishida in [12], where they prove the solution, which is not necessarily spherically symmetric, converges to the corresponding stationary solution as time tends to infinity in the exterior domain of $\mathbb{R}^3$ under the smallness assumptions on the initial data and the external force. For a large initial data, Itaya in [6] shows a time global existence of the spherically symmetric solution to (1.1) on a bounded annulus domain. After this research, many results are obtained for the spherically symmetric solution to (1.1) on a bounded annulus domain with large initial data. For example, Matsumura in [11] considers isothermal flow and proves that the stationary solution is time asymptotically stable for an arbitrary external force. Here, he also obtains a exponential decay rate.

For the spherically symmetric problem in an unbounded exterior domain, Jiang in [7] shows the time global existence of the solution to (1.1) without the external forces, i.e., $f \equiv 0$. In the paper [7], the asymptotic state is not obtained completely. Precisely, it shows that, for $n = 3$, $\|u(t)\|_{L^{2j}} \to 0$ as $t \to \infty$, where $j$ is an arbitrarily fixed integer greater than or equal to 2. Thus, the asymptotic behavior for this problem is remained open. This open problem is solved by Nakamura, Nishibata and Yanagi in [16] for the isentropic flow. In the present paper, we consider the same problem for the ideal polytropic gas to solve this problem.

**Notation.** For a non-negative integer $l \geq 0$, $H^l(\Omega)$ denotes the $l$-th order Sobolev space over $\Omega$ in the $L^2$ sense with the norm $\|\cdot\|_l$. We note $H^0 = L^2$ and $\|\cdot\| := \|\cdot\|_0$. For $\alpha \in (0,1)$, $\mathcal{B}^\alpha(\Omega)$ denotes the space of bounded functions over $\Omega$ which have the uniform Hölder continuity with exponent $\alpha$. For an integer $k$, $\mathcal{B}^{k+\alpha}(\Omega)$ denotes the space of the functions satisfying $\partial_x^i u \in \mathcal{B}^\alpha(\Omega)$ for an arbitrary integer $i \in [0,k]$. For a domain $Q_T \subseteq [0,\infty) \times [0,T]$, $\mathcal{B}^{\alpha,\beta}(Q_T)$ denotes the space of the uniform Hölder continuous functions with the Hölder exponents $\alpha$ and $\beta$ with respect to $x$ and $t$, respectively. For integers $k$ and $l$, $\mathcal{B}^{k+\alpha,l+\beta}(Q_T)$ denotes the space of the functions satisfying $\partial_x^i u, \partial_t^j u \in \mathcal{B}^{\alpha,\beta}(Q_T)$ for arbitrary integers $i \in [0,k]$ and $j \in [0,l]$.

# 2 Local existence in the Lagrangian coordinate

## 2.1 The System in the Lagrangian coordinate

To prove Theorem 1.1, we derive the a priori estimate of the solution by employing the energy method. To this end, it is convenient to transform the equation (1.5) in the Eulerian coordinate into that in the Lagrangian coordinate. The transformation from the Eulerian coordinate $(r,t)$ to the Lagrangian coordinate $(x,t)$ is executed by the transformation

$$x = \int_1^r s^{n-1}\rho(s,t)\,ds, \quad r_x = \frac{v}{r^{n-1}}, \quad r_t = u, \tag{2.1}$$

where $v := 1/\rho$ is the specific volume. Using (2.1), we deduce the system (1.5) to

$$v_t = (r^{n-1}u)_x, \tag{2.2a}$$

$$u_t = \mu r^{n-1}\left(\frac{(r^{n-1}u)_x}{v}\right)_x - r^{n-1}p(v,\theta)_x - U_r, \tag{2.2b}$$

$$c_V\theta_t = \kappa\left(\frac{r^{2n-2}\theta_x}{v}\right)_x - p(v,\theta)(r^{n-1}u)_x + \mu\frac{(r^{n-1}u)_x^2}{v} - 2(n-1)\mu_1(r^{n-2}u^2)_x, \tag{2.2c}$$

where $p(v,\theta) := Rv^{-1}\theta$. The initial and the boundary conditions are

$$(v,u,\theta)(x,0) = (v_0,u_0,\theta_0)(x), \quad v_0 := 1/\rho_0, \tag{2.3}$$

$$u(0,t) = 0, \quad \theta_x(0,t) = 0. \tag{2.4}$$

Owing to (1.8) and (1.9), the initial data $(v_0,u_0,\theta_0)$ satisfies

$$\inf_{x\in[0,\infty)} v_0(x) > 0, \quad \inf_{x\in[0,\infty)} \theta_0(x) > 0, \tag{2.5}$$

$$\lim_{x\to\infty}(v_0,u_0,\theta_0)(x) = (v_+,0,\theta_+), \quad v_+ := 1/\rho_+. \tag{2.6}$$

Since the variable $r = r(x,t)$ is a function of $(x,t)$, the stationary solution $\tilde{v}(r) := 1/\tilde{\rho}(r)$ depends on $(x,t)$ in the Lagrangian coordinate. Thus, let

$$\tilde{v}_0(x) := 1/\tilde{\rho}(r_0(x)), \quad r_0(x) := r(x,0).$$

It is also assumed that the initial data is compatible with the boundary data (2.4), that is, (1.10) holds. The stability theorem of the stationary solution $(\tilde{v},\tilde{u},\tilde{\theta})$ in the Lagrangian coordinate is stated in the next theorem.

**Theorem 2.1.** *Suppose that the initial data $(v_0,u_0,\theta_0)$ satisfies*

$$v_0 - \tilde{v}_0,\ u_0,\ \theta_0 - \theta_+,\ r_0^{n-1}(v_0-\tilde{v}_0)_x,\ r_0^{n-1}u_{0x},\ r_0^{n-1}\theta_{0x} \in L^2(0,\infty). \tag{2.7}$$

*Then there exists a constant $\delta > 0$ such that if the external force $U_r$ satisfies $U_r \in C^1[1,\infty)$ and (1.16), then the initial boundary value problem (2.2), (2.3) and (2.4) has a unique global solution $(v,u,\theta)$ satisfying*

$$v - \tilde{v},\ u,\ \theta - \theta_+,\ r^{n-1}(v-\tilde{v})_x,\ r^{n-1}u_x,\ r^{n-1}\theta_x \in C([0,\infty);L^2(0,\infty)). \tag{2.8}$$

*Moreover the solution $(v,u,\theta)$ converges to the stationary solution $(\tilde{v},0,\theta_+)$ as time $t$ tends to infinity:*

$$\lim_{t\to\infty}\sup_{x\in(0,\infty)}|(v(x,t)-\tilde{v}(r(x,t)),u(x,t),\theta(x,t)-\theta_+)| = 0. \tag{2.9}$$

The initial condition (2.7) holds owing to (1.15a). Because several coefficients in (2.2) are unbounded over the domain $(0,\infty)$, we first consider the approximate problem restricted in the bounded interval $(0,m)$ for a positive integer $m$. Following ideas in [2, 7], we define the "cut-off-function" $\phi_m \in C^3[0,\infty)$ by

$$\phi_m(x) := \begin{cases} 1, & \text{for} \quad 0 \le x \le \dfrac{m}{2}, \\ 0, & \text{for} \quad m \le x, \end{cases} \tag{2.10}$$

$$0 \le \phi_m(x) \le 1, \quad |\partial_x^i \phi_m(x)| \le \frac{C}{m^i} \quad (i=1,2,3), \quad \text{for} \quad \frac{m}{2} \le x \le m,$$

where $m$ is an arbitrary positive integer. We consider the equations for an unknown function $(v_m, u_m, \theta_m)$ in the bounded domain $(0,m)$:

$$v_{mt} = (r_m^{n-1} u_m)_x, \tag{2.11a}$$

$$u_{mt} = \mu r_m^{n-1}\left(\frac{(r_m^{n-1}u_m)_x}{v_m}\right)_x - r_m^{n-1} p(v_m,\theta_m)_x - U_r, \tag{2.11b}$$

$$c_V \theta_{mt} = \kappa\left(\frac{r_m^{2n-2}\theta_{mx}}{v_m}\right)_x - p(v_m,\theta_m)(r_m^{n-1}u_m)_x + \mu\frac{(r_m^{n-1}u_m)_x^2}{v_m} - 2(n-1)\mu_1 (r_m^{n-2}u_m^2)_x \tag{2.11c}$$

with the initial and the boundary conditions

$$(v_m, u_m, \theta_m)(x,0) = (v_{m0}, u_{m0}, \theta_{m0})(x), \tag{2.12}$$

$$u_m(0,t) = u_m(m,t) = 0, \quad \theta_{mx}(0,t) = \theta_{mx}(m,t) = 0, \tag{2.13}$$

where the initial data $(v_{m0}, u_{m0}, \theta_{m0})$ is defined by using $\phi_m$ as

$$\begin{aligned} v_{m0}(x) &:= (v_0(x) - \tilde{v}_0(x))\phi_m(x) + \tilde{v}_0(x), \\ u_{m0}(x) &:= u_0(x)\phi_m(x), \\ \theta_{m0}(x) &:= (\theta_0(x) - \theta_+)\phi_m(x) + \theta_+. \end{aligned}$$

In the equations (2.11), the functions $r_m$ and $r_{m0}$ are given by

$$r_m(x,t) := \left\{1 + n\int_0^x v_m(y,t)\,dy\right\}^{1/n}, \qquad r_{m0}(x) := \left\{1 + n\int_0^x v_{m0}(y)\,dy\right\}^{1/n}.$$

In order to state the local existence theorem precisely, we define the function space:

$$\begin{aligned} &X^m_{D,\bar{v},\underline{v},\bar{\theta},\underline{\theta}}(0,T) := \\ &\quad \big\{(v,u,\theta) \mid (v-\tilde{v}, u, \theta-\theta_+) \in C^0([0,T]; H^1(0,m)), \ \ u_x,\, \theta_x \in L^2(0,T; H^1(0,m)), \\ &\qquad\quad \|(v-\tilde{v}, u, \theta-\theta_+)(t)\|_{1,r,m} \le D \ \text{ for } \ t \in [0,T], \\ &\qquad\quad \underline{v} \le v(x,t) \le \bar{v}, \ \underline{\theta} \le \theta(x,t) \le \bar{\theta} \ \text{ for } \ (x,t) \in [0,m]\times[0,T]\big\} \end{aligned}$$

for constants $D > 0$, $\bar{v} > \underline{v} > 0$, $\bar{\theta} > \underline{\theta} > 0$ and $T > 0$. The norm $\|\cdot\|_{1,r,m}$ is given by

$$\|(v-\tilde{v}, u, \theta-\theta_+)\|_{1,r,m} := \|(v-\tilde{v},\ u,\ \theta-\theta_+,\ r^{n-1}(v-\tilde{v})_x,\ r^{n-1}u_x,\ r^{n-1}\theta_x)\|_{L^2(0,m)}.$$

A time local solution to the initial boundary value problem (2.11), (2.12) and (2.13) is established by employing the standard iteration method.

**Proposition 2.2.** *For arbitrary constants $D > 0$, $\bar{v} > \underline{v} > 0$ and $\bar{\theta} > \underline{\theta} > 0$, there exists a positive constant $T = T(D, \bar{v}, \underline{v}, \bar{\theta}, \underline{\theta})$ such that if the initial data $(v_{m0}, u_{m0}, \theta_{m0})$ satisfies $\|(v_{m0} - \tilde{v}, u_{m0}, \theta_{m0} - \theta_+)\|_{1,r,m} \le D$, $\underline{v} \le v_{m0}(x) \le \bar{v}$ and $\underline{\theta} \le \theta_{m0}(x) \le \bar{\theta}$, then the problem (2.11), (2.12) and (2.13) has a unique solution $(v_m, u_m, \theta_m) \in X^m_{2D,2\bar{v},\underline{v}/2,2\bar{\theta},\underline{\theta}/2}(0,T)$.*

## 2.2 A priori estimate

In this subsection, we discuss the a priori estimate, uniformly in $m$, in $H^1$-Sobolev space for the solution $(v_m, u_m, \theta_m) \in X^m_{D,\bar{v},\underline{v},\underline{\theta}}(0,T)$. This estimate gives the existence of a solution to the problem (2.2), (2.3) and (2.4) over the unbounded domain $(0,\infty)$. Hereafter in this subsection, for simplicity, we omit the subscript $m$ and denote $(v_m, u_m, \theta_m, r_m)$ by $(v, u, \theta, r)$.

We define the energy form $\mathcal{E}$ as

$$\mathcal{E} := \frac{1}{2}u^2 + \Psi(v,\tilde{v}) + c_V\theta_+\left(\frac{\theta}{\theta_+} - 1 - \log\frac{\theta}{\theta_+}\right), \tag{2.14}$$

$$\Psi(v,\tilde{v}) := p(\tilde{v},\theta_+)(v-\tilde{v}) - \varphi(v,\tilde{v}), \tag{2.15}$$

$$\varphi(v,\tilde{v}) := \int_v^{\tilde{v}} p(\xi,\theta_+)\,d\xi = R\theta_+\log\frac{v}{\tilde{v}}. \tag{2.16}$$

The following lemmas are proved by using the standard energy method. For details, see [15].

**Lemma 2.3.** *For a solution $(v,u,\theta) \in X^m_{D,\bar{v},\underline{v},\bar{\theta},\underline{\theta}}(0,T)$ to (2.11), (2.12) and (2.13),*

$$\int_0^m \mathcal{E}(t)\,dx + c\int_0^t\int_0^m \frac{v}{\theta r^2}u^2 + \frac{r^{2n-2}}{v\theta}u_x^2 + \frac{r^{2n-2}}{v\theta^2}\theta_x^2\,dx\,d\tau \le \int_0^m \mathcal{E}(0)\,dx, \tag{2.17}$$

*where $c$ is a positive constant depending only on the initial data.*

**Lemma 2.4.** *For a solution $(v,u,\theta) \in X^m_{D,\bar{v},\underline{v},\bar{\theta},\underline{\theta}}(0,T)$ to (2.11), (2.12) and (2.13),*

$$\int_0^m \frac{u^4 + (\theta-\theta_+)^2}{r^{2n-2}}\,dx + \int_0^t\int_0^m u^2u_x^2 + \theta_x^2\,dx\,d\tau \le C_{\bar{v},\underline{v}}, \tag{2.18}$$

*where $C_{\bar{v},\underline{v}}$ is a positive constant depending only on $\bar{v}$, $\underline{v}$ and the initial data.*

**Lemma 2.5.** *For a solution $(v,u,\theta) \in X^m_{D,\bar{v},\underline{v},\bar{\theta},\underline{\theta}}(0,T)$ to (2.11), (2.12) and (2.13),*

$$\int_0^m \varphi_x^2\,dx + \int_0^t\int_0^m (1+\theta)\varphi_x^2\,dx\,d\tau \le C_{\bar{v},\underline{v}}, \tag{2.19}$$

$$\int_0^m u_x^2\,dx + \int_0^t\int_0^m r^{2n-2}u_{xx}^2\,dx\,d\tau \le C_{\bar{v},\underline{v}}(\bar{\theta}+1), \tag{2.20}$$

$$\int_0^m \theta_x^2\,dx + \int_0^t\int_0^m r^{2n-2}\theta_{xx}^2\,dx\,d\tau \le C_{\bar{v},\underline{v}}(\bar{\theta}+1)^2. \tag{2.21}$$

Using the estimates (2.20) and (2.21), we show the uniform positive bound of $u$ and $\theta$ in the next lemma. Thus, we see that the estimates (2.20) and (2.21) are independent of $\bar{\theta}$.

**Lemma 2.6.** *For a solution $(v,u,\theta) \in X^m_{D,\bar{v},\underline{v},\bar{\theta},\underline{\theta}}(0,T)$ to (2.11), (2.12) and (2.13),*

$$|u(x,t)| \leq C_{\bar{v},\underline{v}}, \tag{2.22}$$

$$0 < ce^{-C_{\bar{v},\underline{v}}(1+t)} \leq \theta(x,t) \leq C_{\bar{v},\underline{v}} \tag{2.23}$$

*for $(x,t) \in [0,m] \times [0,T]$.*

**Lemma 2.7.** *For a solution $(v,u,\theta) \in X^m_{D,\bar{v},\underline{v},\bar{\theta},\underline{\theta}}(0,T)$ to (2.11), (2.12) and (2.13),*

$$\int_0^m r^{2n-2}\varphi_x^2 + r^{2n-2}u_x^2 + r^{2n-2}\theta_x^2 \, dx \leq C_{\bar{v},\underline{v}}. \tag{2.24}$$

Utilizing the estimates obtained in the above lemmas and letting $m \to \infty$, we have the local solution $(v,u,\theta)$ to the problem (2.2), (2.3) and (2.4) satisfying

$$v - \tilde{v},\ u,\ \theta - \theta_+,\ r^{n-1}(v-\tilde{v})_x,\ r^{n-1}u_x,\ r^{n-1}\theta_x \in C([0,T];L^2(0,\infty)),$$
$$\varphi_x,\ r^{n-1}u_x,\ r^{n-1}\theta_x,\ r^{n-1}u_{xx},\ r^{2n-2}\theta_{xx} \in L^2(0,T;L^2(0,\infty)).$$

Moreover, we have the energy estimates over an unbounded domain $(0,\infty)$ as

$$\int_0^\infty \mathcal{E}\,dx + \int_0^t\int_0^\infty \frac{v}{\theta r^2}u^2 + \frac{r^{2n-2}}{v\theta}u_x^2 + \frac{r^{2n-2}}{v\theta^2}\theta_x^2 \, dx\,d\tau \leq C, \tag{2.25a}$$

$$\int_0^\infty r^{2n-2}(v-\tilde{v})_x^2 + r^{2n-2}u_x^2 + r^{2n-2}\theta_x^2 \, dx + \int_0^t\int_0^\infty \varphi_x^2 + r^{2n-2}u_{xx}^2 + r^{4n-4}\theta_{xx}^2 \, dx\,d\tau \leq C_{\bar{v},\underline{v}}. \tag{2.25b}$$

# 3 Asymptotic stability of the stationary solution in Lagrangian coordinate

## 3.1 Positive bound of the specific volume

In this subsection, we derive the pointwise positive bound of the specific volume $v(x,t)$ uniformly in $t$. We prove this bound by using the basic estimate (2.25a) and the representation formula of $v(x,t)$.

**Proposition 3.1.** *Let $(v,u,\theta) \in X^\infty_{D,\bar{v},\underline{v},\bar{\theta},\underline{\theta}}(0,T)$ be a solution to the problem (2.2), (2.3) and (2.4). Then there exist positive constants $c$ and $C$ depending only on the initial data such that if the external force $U_r$ satisfies*

$$-\delta_D \leq U_r(r), \quad \delta_D := \frac{1}{2}\left(\sqrt{2}D + v_+e^{\frac{M}{R\theta_+}}\right)^{-1}, \tag{3.1}$$

*where $M$ is a positive constant satisfying* $\sup_{r\geq 1} U(r) \leq M$, *then the specific volume* $v(x,t)$ *satisfies*

$$c \leq v(x,t) \leq C \quad \textit{for} \quad (x,t) \in [0,\infty) \times [0,T]. \tag{3.2}$$

To prove (3.2), we derive the representation formula of the specific volume $v(x,t)$. To this end, we define the "cut-off-function" $\eta(x)$:

$$\eta(x) := \begin{cases} 1, & \text{for } 0 \leq x \leq k\varepsilon, \\ k+1-\dfrac{x}{\varepsilon}, & \text{for } k\varepsilon \leq x \leq (k+1)\varepsilon, \\ 0, & \text{for } (k+1)\varepsilon \leq x \end{cases} \tag{3.3}$$

for $\varepsilon > 0$ and $k = 1, 2, \ldots$.

**Lemma 3.2.** *Suppose that the same assumptions as in Proposition 3.1 hold. Then the specific volume $v(x,t)$ is expressed by the formula*

$$v(x,t) = \frac{v_0(x) + \frac{R}{\mu}\int_0^t A_\varepsilon(x,\tau)B_\varepsilon(x,\tau)\theta(x,\tau)\,d\tau}{A_\varepsilon(x,t)B_\varepsilon(x,t)} \tag{3.4}$$

*for $x \in [(k-1)\varepsilon, k\varepsilon)$ and $t \in [0,T]$, where*

$$A_\varepsilon(x,t) := A_\varepsilon^0(t)\, A_\varepsilon^1(x,t)\, A_\varepsilon^2(x,t),$$

$$A_\varepsilon^0(t) := \exp\left(\frac{R}{\mu\varepsilon}\int_0^t\int_{k\varepsilon}^{(k+1)\varepsilon} \frac{\theta}{v}\,dx\,d\tau\right), \quad A_\varepsilon^1(x,t) := \exp\left(\frac{n-1}{\mu}\int_0^t\int_x^\infty \frac{u^2}{r^n}\eta\,dx\,d\tau\right),$$

$$A_\varepsilon^2(x,t) := \exp\left(\frac{1}{\mu}\int_0^t\int_x^\infty \frac{U_r}{r^{n-1}}\eta\,dx\,d\tau\right),$$

$$B_\varepsilon(x,t) := \exp\left(\frac{1}{\mu}\int_x^\infty \left(\frac{u}{r^{n-1}} - \frac{u_0}{r_0^{n-1}}\right)\eta\,dx - \frac{1}{\varepsilon}\int_{k\varepsilon}^{(k+1)\varepsilon} \log\frac{v}{v_0}\,dx\right). \tag{3.5}$$

Owing to the estimate (2.25a) with the aid of the Jensen inequality, we have the estimates

$$c_\varepsilon \leq \int_a^{a+\varepsilon} v(x,t)\,dx \leq C_\varepsilon, \quad c_\varepsilon \leq \int_a^{a+\varepsilon} \theta(x,t)\,dx \leq C_\varepsilon \tag{3.6}$$

for an arbitrary constant $a \geq 0$. Substituting (3.6) in (3.4), we obtain the desired estimate (3.2). For details, readers are referred to [15].

Substituting the pointwise estimate (3.2) in (2.25b), we obtain a uniform a priori estimate:

$$\int_0^\infty (v-\tilde{v})^2 + u^2 + (\theta-\theta_+)^2 + r^{2n-2}(v-\tilde{v})_x^2 + r^{2n-2}u_x^2 + r^{2n-2}\theta_x^2\,dx$$
$$+ \int_0^t\int_0^\infty \frac{u^2}{r^2} + \varphi_x^2 + r^{2n-2}u_x^2 + r^{2n-2}\theta_x^2 + r^{2n-2}u_{xx}^2 + r^{4n-4}\theta_{xx}^2\,dx\,d\tau \leq C. \tag{3.7}$$

The estimate (3.7) yields the existence of the solution to the problem (2.2), (2.3) and (2.4) globally in time by the standard continuation argument. Moreover, we see the convergence (2.9) holds. Finally, by employing the Schauder theory for the parabolic equations, we have the estimate of the solution in the Hölder space. By virtue of the Hölder continuity of the solution, we translate Theorem 2.1 into that in the Eulerian coordinate. These procedures show the convergence (1.18).

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